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論 文

[2018] The One-Point Integration Rule in Nonlinear Finite Element Analysis

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1. INTRODUCTION

Analysis of the localization,the phenomenon that large strains concentrate into a thin band without affecting the other portions of the structure, is thought to be a major challenge in computational mechanics. Many methods in finite element with a discontinuous field have been proposed to successfully solve this problem [1][2].No matter which method is used, the one point quadrature scheme is considered to be the most efficient one.

It has been known for long time that one-point quadrature scheme provides tremendous benefits in nonlinear algorithms because the number of evaluations of the semidiscretized gradient operator, commonly know as the [B] matrix, and the constitutive equations, is reduced substantially.

However, the use of one-point quadrature scheme results in certain hourglass modes, or zero-energy modes. If a mesh is consistent with a global pattern of these modes, they will quickly dominate and destroy the solution. Some methods have been proposed to deal with this phenomenon. Those include the one proposed by Kosloff and Frazier[3], Flanagan and Belytschko[4]. The method proposed by Kosloff and Frazier has been thought to be the most effective and the easiest to understand particularly for rectilinear elements.

In the paper written by Kosloff and Frazier, their hourglass control method was proven to be correct and effective in the elastic analysis, but nothing was included in the scope of nonlinear analysis. In this paper, a method to control the hourglass modes in nonlinear calculation is developed. A modified scheme is proposed to obtain the accurate response of the structure in nonlinear region. The discussions of this paper are confined within the rectangular linear element. The research for a more general case will be presented in future publications.

2. HOURGLASS CONTROL METHOD[3][5]

The strain energy in an element U_e can be expressed as:

$$U_e = \frac{1}{2}\{u\}^T[K]\{u\} = \frac{1}{2}\{u\}^T\int_{v_e}[B]^T[D][B]dv\,\{u\} = \frac{1}{2}\int_{v_e}\{\epsilon\}^T[D]\{\epsilon\}\,dv \qquad (1)$$

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where {u} is the displacement vector, [K] the element stiffness matrix, [D] the material constitutive matrix, [B] the strain matrix and $\{\varepsilon\}$ the strain vector.

Fig.1 shows the eight independent displacement modes of a four-node plane element. Modes 7 and 8 are bending modes. Since σ_x, σ_y and τ_{xy} are always zero at their center, according to **eq.(1)**, $\mathbf{U}_e$ becomes zero when using one-point (center point) integration.

To control these zero-energy modes, restraints are introduced, but at the same time without influence on the element's response to the already existing modes that have already been working well. For simplicity, consider only the x-direction nodal displacement $\{\mathbf{U}_x\}$ of the element shown in **Fig.1** . For modes 1 and 8, $\{\mathbf{U}_x\} = 0$, and an arbitrary combination of modes 2 through 7 is

$$\{\mathbf{u}_x\} = \{\mathbf{u}_2\} + \{\mathbf{u}_3\} + \{\mathbf{u}_4\} + \{\mathbf{u}_5\} + \{\mathbf{u}_6\} + \{\mathbf{u}_7\}$$

$$= a_2 \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + a_3 \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix} + a_4 \begin{Bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{Bmatrix} + a_5 \begin{Bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{Bmatrix} + a_6 \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix} + a_7 \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix} \tag{2}$$

To provide mode 7 with the stiffness which it lacks under one-point quadrature, add $[\mathrm{K}]_7$ to the element stiffness matrix,where

$$[K]_7 = \{\mathbf{u}_7\}\{\mathbf{u}_7\}^T \tag{3}$$

By the same way, $[\mathrm{K}]_8$ can be obtained to prevent the mode 8 occurring.

It is possible to choose values of a_7 and a_8 such that a rectangular element displays the exact strain energy caused by the pure bending. So in this case

$$a_7 = (\frac{E}{12}\frac{B}{A})^{\frac{1}{2}}, \qquad a_8 = (\frac{E}{12}\frac{A}{B})^{\frac{1}{2}} \tag{4}$$

where A,B are the length of the element (**Fig. 1**) and E the young's modulus. Finally, the hourglass control matrix is obtained as:

$$[K]_h = [K]_7 + [K]_8 \tag{5}$$

So the element stiffness matrix can be obtained as:

$$[K] = [K]_{Q4} + [K]_h \tag{6}$$

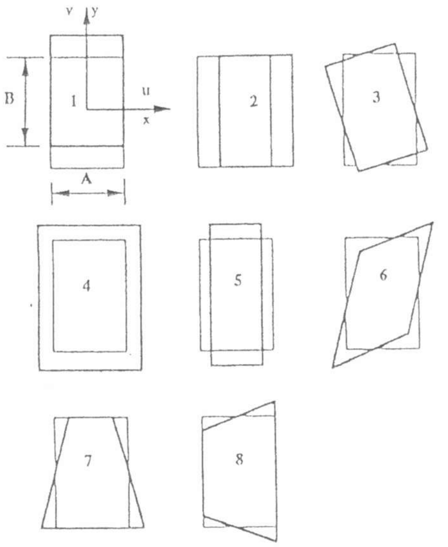

Fig.1 Independent displacement modes of a four-node plane element

where $[K\ ]_{Q4}$ is the retilinear element stiffness matrix calculated by one-point integration.

3. ONE-POINT INTEGRATION RULE IN NONLINEAR CALCULATION

3.1 Tangential Stiffness Matrix

It has been proven by Kosloff and Frazier, that using **eqs.(5)** and **(6)** in elastic region, the result of the element stiffness matrix turns out to be the same as that obtained by using the Q6 incompatible element[6]. In inelastic region, to obtain the tangential stiffness matrix, different values of E_x and E_y, which are corresponding to the up-dated material constitutive matrix $[\mathbf{D}_{ep}]$ of material, should be substituted into **eq.(4)**, that is

$$a_7 = (\frac{E_x}{12}\frac{B}{A})^{\frac{1}{2}}, \qquad a_8 = (\frac{E_y}{12}\frac{A}{B})^{\frac{1}{2}} \tag{7}$$

By setting the stress vector $\{\sigma\} = (\sigma_x, 0, 0,)^T$,and substituting it into the up-dated stress-strain relation $[\mathbf{D_{ep}}]\{\varepsilon\} = \{\sigma\}$, $\sigma_x = E_x\varepsilon_x$ can be obtained and E_x can be evaluated. By the same way, by setting $\{\sigma\} = (0, \sigma_y, 0)^T$,$E_y$ can be evaluated.

3.2 Strain and Stress Calculation

When **eq.(6)** is used to evaluate the stiffness, and the equation $\{\varepsilon\} = [B]\{\mathbf{u}\}$ is used to calculate the strain,the strain is correct only at the center point, because the strain do not include the one corresponding to modes 7 and 8.

Since $\{\mathbf{u}\}$ is known, it is possible to calculate the strain and stress caused by hourglass modes (pure bending modes). For simplicity, first consider the x direction bending mode. Suppose the element has the nodal displacement along x direction as shown in **Fig.2**. The total nodal displacement can be divided into three modes, mode 2, 3 and mode 7. So the strain caused by mode 7 at the point at y distance from the x axis will be

$$\varepsilon_{x7} = \frac{y}{B}\left(\frac{d_4 - d_2}{2} - \frac{d_3 - d_1}{2}\right), \qquad \varepsilon_{y7} = 0 \tag{8}$$

where d_i (i = 1,2,3,4) is the nodal displacement along x direction.

The stress caused by the mode 7 are

$$\sigma_{x7} = E_x\varepsilon_{x7}, \qquad \sigma_{y7} = 0 \tag{9}$$

In the same way,σ_{x8}, σ_{y8} can be evaluated. By adding $\sigma_{x7}, \sigma_{y7}, \sigma_{x8}$ and σ_{y8} into the stress at center point,the correct stress at any point of the element can be evaluated. These stresses are shown to be the same as those obtained by Q6 element.

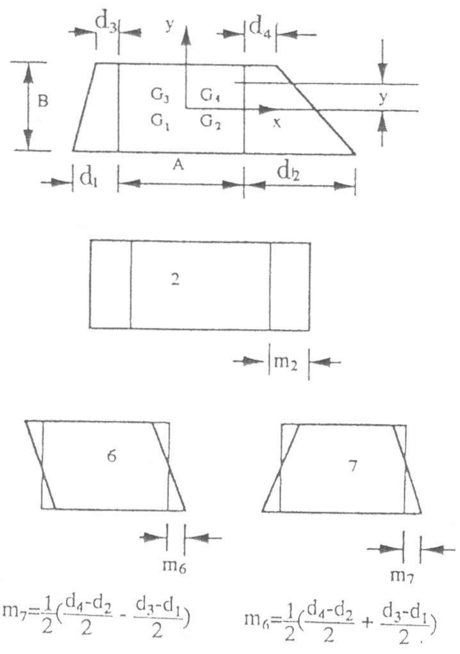

Fig.2 Strain caused by hourglass mode

3.3 The Residual Force

From **eq.(6)**, we can write

$$([K]_{Q4} + [K]_h)\{\mathbf{u}\} = \{\mathbf{R}\} \tag{10}$$

where $\{R\}$ is the element nodal force.

So obviously the equivalent nodal force caused by mode 7 and 8 are

$$\{\mathbf{F}_h\} = [K]_h\{\mathbf{u}\} \tag{11}$$

so the equation to evaluate the residual force is

$$\{\mathbf{Q}\} = \{\mathbf{R}\} - \{\mathbf{F}\} - \{\mathbf{F}_h\} \tag{12}$$

where $\{\mathbf{F}\} = \int_{v_e}[\mathbf{B}]^{\mathbf{T}}[\mathbf{D}][\mathbf{B}]\mathbf{dv}$.

4. MODIFIED METHOD

Because of the similarity between the hourglass control method and the Q6 incompatible element, the results obtained by using Q6 incompatible element naturally are taken as a checking criterion in nonlinear calculation when the mentioned hourglass control method is used.

At the center of the element, the stress state, which is the average stress of the whole element, is used to estimate the response of the element. So it can be reasoned out that, when the stronger are the bending mode 7 and 8 (which means the stress along x and y directions change a lot from one side of the element to the other side of the element, the extreme case is the pure bending of one element), the larger are the difference between the element response obtained by using the hourglass control method and that obtained by using Q6 incompatible element method. This will be shown in numerical examples latter in this paper.

To solve the problem of this inaccurate response of the element when one-point integration method, prescribed from section 3.1 to 3.3, is used in nonlinear analysis, a modified method is proposed as follows:

1. Use **eq.(6)** to evaluate the element stiffness matrix by one-point integration rule.
2. Use section 3.2 to calculate the stress at the point of 2 × 2 Gauss points.
3. Use the stress at 2 × 2 Gauss points to estimate the nonlinear response of the elements.
4. According to the equation $\{\mathbf{F}\} = \int_{v_e} [\mathbf{B}]^{\mathrm{T}} \{\sigma\}\, \mathbf{dv}$,use the stress at 2 × 2 Gauss points to calculate the equivalent nodal forces of the element.

In step 3, when the 2 × 2 Gauss points are used to estimate the nonlinear response of the element, different values of $[\mathbf{D}_{eq}]$ of these four points can be obtained, by averaging these four $[\mathbf{D}_{eq}]$ and using **eq.(7)** to calculate the element stiffness [K], the convergence rate will become higher when tangential stiffness method (the element stiffness are recomputed during each iteration of load increment) and combined algorithm (the element stiffness are recomputed for first iteration of each load increment only) are used in nonlinear analysis.

Compared with the Q4 element, obviously,the modified method can reduce time greatly in the stiffness matrix calculation and has the benefit of accuracy of evaluating the bending response of the element. Compared with the Q6 incompatible element, it can reduce time greatly in calculating the stiffness matrix and the stiffness matrix condensation and at the same time it has the same accuracy response in both linear and nonlinear analysis.

5. NUMERICAL EXAMPLE

In this section, arc-length method[7] is used in the two-dimensional nonlinear finite element program in order to obtain the post peak response of the structure. Before the peak point, the tangential stiffness method is used, and after the peak point, the initial stiffness method is used. In this section OPI method denotes one-point integration with hourglass control, MOPI method the modified method described in section 4, Q4 element the 4-node quadrilateral element, Q6 element the incompatible element .

A beam is subjected to a transverse external force at the middle of the span. Three types of meshes used to stimulate the response of the beam, are shown in **Fig.3**. **Fig.4** shows the material properties with young modulus $E_1 = 2.1 \times 10^6 kg/cm^2$, plastic modulus $E_2 = 0$ and the uniaxial yield stress $\sigma_y = 2500kg/cm^2$. The material is assumed to follow J_2 theory.

Fig.5 to **Fig.7** show the load versus displacement curves by using mesh(a), mesh(b) and mesh(c) respectively.

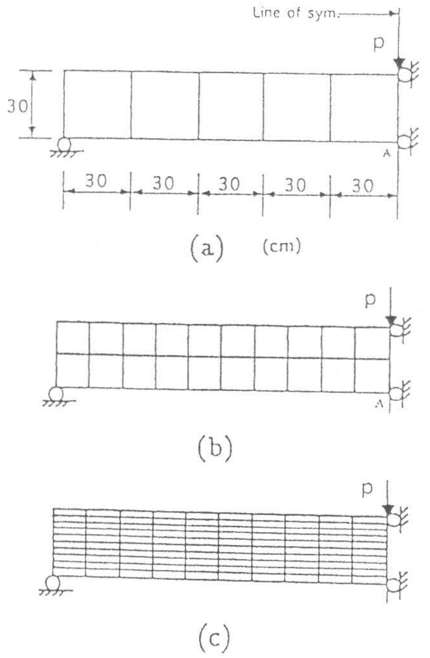

Fig.3 Structure dimension and FEM meshs

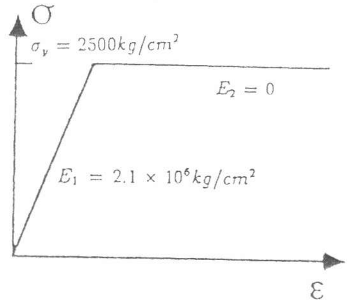

Fig.4 Stress-strain relation

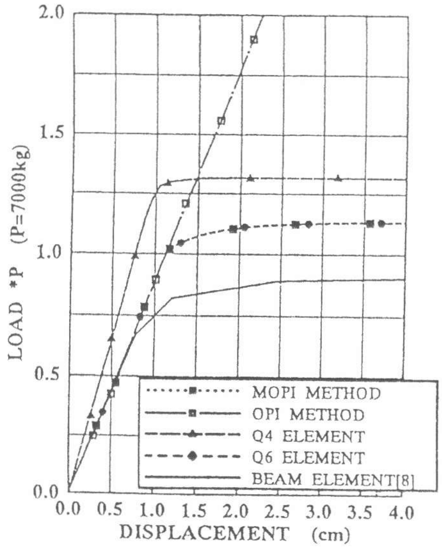

Fig.5 Load-displacement relation for mesh (a)

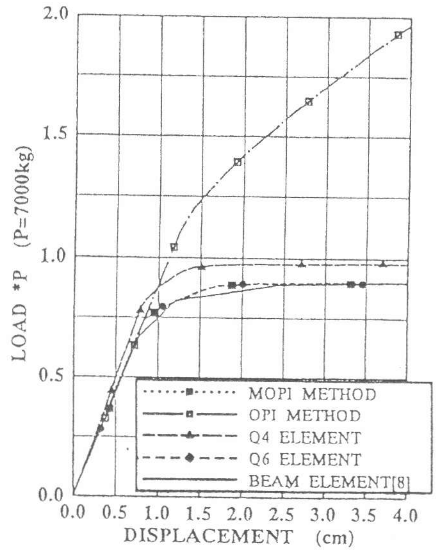

Fig.6 Load-displacement relation for mesh (b)

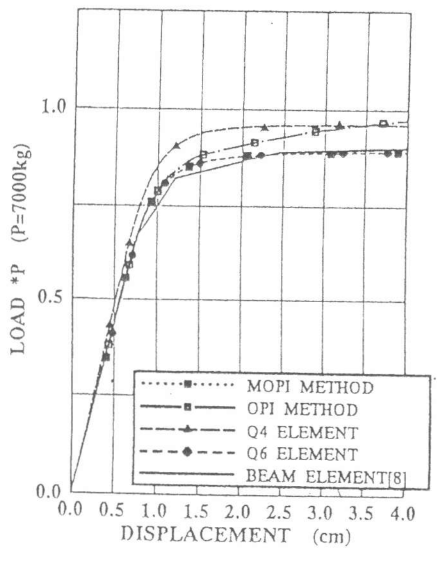

Fig.7 Load-displacement relation for mesh (c)

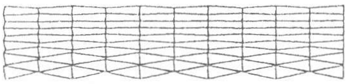

Fig.8 Deformed mesh (c) with-.out hourglass control

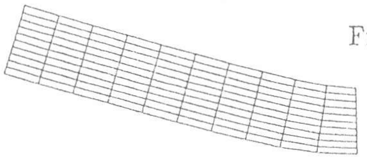

Fig.9 Deformed mesh (c) with hourglass control

When the Q6 element or MOPI method is used, the same results can be obtained as those obtained by OPI method in linear region. OPI method can accurately represent flexural modes of the deformations in shallow structures(like the beam in this example, even for mesh(a) or mesh(b) where fewer elements are used, rather accurate results can be obtained), but Q4 element behaves stiffer. So to analyze a shallow structure in the elastic region, OPI method is a recommendable method, because of saving time, using fewer elements and having accurate results, but it is not in the case of nonlinear analysis. In the nonlinear region, the OPI method gives a bad estimation of the structure response (as shown in **Fig.5** and **Fig.6** , unless a more refined mesh is used (as mesh(c)) .On the other hand, as shown in **Fig.5**, **6**,and**7** , MOPI method can give as accurate results as Q6 element both in linear and nonlinear region and consume much less calculating time . So as nonlinear analysis is involved, MOPI method is a recommendable method.

Without hourglass control,the nonlinear calculation can seldom be carried on because of the affection of the hourglass modes. **Fig.8** shows the deformed mesh(c) without hourglass control when very small external force is applied onto the structure. **Fig.9** shows the deformed mesh(c) using MOPI method.

8. CONCLUSIONS

A method was developed to eliminate hourglass instabilities in four-node rectangular elements in nonlinear problems. A modified scheme has been presented to obtain accurate nonlinear element flexural response. Through numerical calculations, it has been proven that this modified scheme can give as accurate results as the Q6 incompatible element[5] and consume less time.

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